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# On the Pluricanonical Systems of Algebraic Manifolds (A SYMPOSIUM ON COMPLEX MANIFOLDS)

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On the pluricanonical systems of algebraic manifolds.

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Any algebraic manifold is assumed to be connected, complete, non-singular and defined over the complex number field $\mathbb{C}$. Let $K_M$ be the canonical line bundle of an algebraic manifold $M$. If $P_m(M) = \dim_{\mathbb{C}} H^0(M, \underline{O}(mK_M))$ is positive for a positive integer $m$, we can define a rational mapping

$$\begin{array}{cccc} \Phi_{mK} : & M & \longrightarrow & \mathbb{P}^N \\ & \Psi & & \Psi \\ & z & \longmapsto & (\varphi_0(z):\varphi_1(z):\cdots:\varphi_N(z)), \end{array}$$

where $\{\varphi_0, \varphi_1, \ldots, \varphi_N\}$ is a basis of the vector space $H^0(M, \underline{O}(mK_M))$. The rational mapping $\Phi_{mK}$ is called the m-th canonical mapping. We set $\mathbb{N}(M) = \{m > 0 \mid P_m(M) > 0\}$. The Kodaira dimension $\kappa(M)$ of the algebraic manifold $M$ is defined by

$$\kappa(M) = \begin{cases} \max\limits_{m \in \mathbb{N}(M)} \dim \Phi_{mK}(M) & \text{if } \mathbb{N}(M) \neq \emptyset, \\ 0 & \text{if } \mathbb{N}(M) = \emptyset . \end{cases}$$

It is easy to show that, if two algebraic manifolds $M_1$ and $M_2$

are birationally equivalent, then $P_m(M_1) = P_m(M_2)$. Hence for an irreducible complete singular algebraic variety $V$, we define the Kodaira dimension $\kappa(V)$ of $V$ by

$$\kappa(V) = \kappa(V^*)$$

where $V^*$ is a non-singular model of the variety $V$. For the properties of Kodaira dimensions we refer the reader to Iitaka [2] and Ueno [6], [7].

Let $S$ be an algebraic surface, that is, an algebraic manifold of dimension two. A complete curve $C$ in $S$ is called an exceptional curve of the first kind if $C$ is a non-singular rational curve with $C^2 = -1$. If $S$ contains an exceptional curve $C$ of the first kind, there exist a non-singular surface $\hat{S}$ and a birational morphism $\varphi : S \longrightarrow \hat{S}$ such that $\varphi(C)$ is a point $\hat{p}$ and that $\varphi$ induces an isomorphism between $S - C$ and $\hat{S} - \hat{p}$. The following theorem is a corollary to the classification theory of algebraic surfaces.

Theorem. _Let_ $S$ _be an algebraic surface free from exceptional curves of the first kind. Suppose that_ $\kappa(S) \geq 0$. _Then there exist a positive integers_ $d$ _and_ $m_0$ _such that the complete linear system_ $|mdK_S|$ _is free from base points and fixed components if_ $m \geq m_0$.

If $\kappa(S) = 2$, then we can show that $d = 1$ and $m_0 = 4$. The proof can be found in Kodaira [4] and Bombieri [1]. If $\kappa(S) = 0$, then the number $d$ can be taken as a divisor of 12 and

$m_0 = 1$. The proof can be found in Šafarevič et al [5] Chap. VIII. If $\kappa(S) = 1$, then the number $d$ can be taken as a divisor of 86. This fact can be deduced from the canonical bundle formula for elliptic surfaces due to Kodaira [3].

It had not been known whether the above theorem holds for algebraic manifolds of dimension $n \geqq 3$. The main purpose of the present paper is to show that the above theorem does not necessarily hold for an algebraic manifold of dimension $n \geqq 3$. Namely, we shall prove the following :

Main Theorem. For a pair of positive integers $\ell$, $n$ with $0 \leqq \ell \leqq n$, $3 \leqq n$, there exists an algebraic manifold $M$ of dimension $n$ which satisfies the following conditions:

① $\kappa(M) = \ell$,

② For any birationally equivalent non-singular manifold $M^*$ of $M$, if $|mK_{M^*}| \neq \emptyset$, then $|mK_{M^*}|$ has fixed components.

To prove the theorem we shall construct algebraic manifolds which satisfy the above conditions ①, ② using the canonical resolutions of cyclic quotient singularities. For simplicity, in this paper, we shall only consider the quotient singularity by a cyclic group of order 2. It is not dificult to generalize our construction to the case of arbitrary quotient singularities.

§1. Let $M$ be an algebraic manifold and let $\underline{S}^k(\Omega_M^\ell)$ be the k-th symmetric tensor product of the sheaf $\Omega_M^\ell$ of germs of holomorphic $\ell$-forms on $M$. The following lemma is well-known. A proof is found in Ueno [6].

Lemma 1.1 Let $M$ and $M^*$ be algebraic manifolds. Suppose that there exists a surjective rational mapping $f : M \longrightarrow M^*$. Then for any positive integer $k$, $f$ induces an injective linear mapping

$$f^* : H^0(M^*, \underline{S}^k(\Omega^\ell_{M^*})) \longrightarrow H^0(M, \underline{S}^k(\Omega^\ell_M)).$$

Moreover if $f$ is birational, $f^*$ is an isomorphism.

Now we shall consider resolutions of quotient singularities. Let $U$ be an open set in $\mathbb{C}^n$ defined by inequalities :

$$|z_i| < (\varepsilon)^{1/2}, \quad i = 1, 2, \ldots, n.$$

We let $G$ be a group of order $2$ of analytic automorphisms of $U$ generated by the automorphism

$$g : (z_1, z_2, \ldots, z_n) \longmapsto (-z_1, -z_2, \ldots, -z_n).$$

The quotient space $\hat{U} = U/G$ has a singular point $p$ which corresponds to the origin of $\mathbb{C}^n$. A resolution of the singularity of $\hat{U}$ can be given as follows. Let $W_i$, $i = 1, 2, \ldots, n$ be open set of $\mathbb{C}^n$ defined by the inequalities :

$$|(w_i^k)^2 w_i^i| < \varepsilon, \quad k \neq i, \quad |w_i^i| < \varepsilon.$$

We shall construct a complex manifold $W = \bigcup_{i=1}^n W_i$ by identifying $W_{i-1}$ and $W_i$ through the following relations :

$$\begin{cases} w_i^k = w_{i-1}^k / w_{i-1}^i, & k \neq i-1, i, \\ w_i^{i-1} = 1/w_{i-1}^i \end{cases}$$

$$\left\{ w^i_i = (w^i_{i-1})^2 w^{i-1}_{i-1} \; . \right.$$

Let us consider a meromorphic mapping

$$(1.2) \qquad \begin{array}{ccc} T_i : U & \longrightarrow & W_i \\ \cup & & \cup \\ (z_1, z_2, \ldots, z_n) & \longmapsto & \left(\frac{z_1}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, (z_i)^2, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i}\right). \end{array}$$

The meromrophic mappings $T_i$, $i = 1,\ldots,n$ induce a meromorphic mapping $T : \hat{U} \longrightarrow W$. Let $E$ be a submanifold of $W$ defined by the equations :

$$w^i_i = 0 \quad \text{in} \quad W_i, \quad i = 1,2,\ldots,n \; .$$

$E$ is analytically isomorphic to an (n-1)-dimensional complex projective space $\mathbb{P}^{n-1}$. The meromorphic mapping $T$ incudes an isomorphism between $\hat{U} - p$ and $W - E$. Hence we infer that $W$ is a non-singular model of the quotient space $\hat{U} = U/G$. The procedure of resolving the singularity is called the <u>canonical resolution</u>.

Let us consider the G-invariant subspace $H^0(U, \underline{S}^k(\Omega^\ell_U))^G$ of $H^0(U, \underline{S}^k(\Omega^\ell_U))$. Any element $\varphi$ of $H^0(U, \underline{S}^k(\Omega^\ell_U))^G$ gives an element $\hat{\varphi}'$ of $H^0(\hat{U}\text{-}p, \underline{S}^k(\Omega^\ell_{\hat{U}-p}))^G$.
By 1.2 we can easily show that $\hat{\varphi}'$ can be uniquely extended to a meromorphic section $\hat{\varphi}$ of the locally free sheaf $\underline{S}^k(\Omega^\ell_W)$.
By explicit calculations we can prove the following:

Lemma 1.3. ① <u>If</u> $n \geqq 2$, <u>there is a canonical isomorphism</u>

$$\begin{array}{ccc} H^0(U, \underline{O}(mK_U))^G & \xrightarrow{\sim} & H^0(W, \underline{O}(mK_W)) \\ \cup & & \cup \\ \varphi & \longmapsto & \hat{\varphi} \; . \end{array}$$

*Moreover, if* $n \geqq 3$, *any element* $\varphi$ *of* $H^0(U, \underline{O}(mK_U))^G$ *has a zero of order at least* $[\frac{m}{2}]$ *on* $E$ *where* $[\ ]$ *is the Gauss symbol*.

② *The form* $(dz_1)^2 \in H^0(U, \underline{S}^2(\Omega_U^1))^G$ *induces a meromorphic section* $\psi$ *of the sheaf* $\underline{S}^2(\Omega_W^1)$ *which has a pole of order* 1 *on* $E$.

Remark 1.4. If $n = 2$, $(dz_1 \wedge dz_2)^m$ is an element of $H^0(U, \underline{O}(mK_U))^G$ and induces a nowhere vanishing element of $H^0(W, \underline{O}(mK_W))^G$. This is one of the main differences between dimention two and dimension $n \geqq 3$.

§ 2. Main Theorem is a corollary of the following theorem.

Theorem 2.1. *Let* $V$ *be an algebraic manifold of dimension* $n \geqq 3$. *Suppose that* $V$ *has an analytic involution* $g$. *Suppose, moreover,*

① *the involution* $g$ *has at least one fixed point and any fixed manifold of* $g$ *is an isolated point* ;

② *there exists a holomorphic 1-form* $\omega$ *on* $V$ *such that* $\omega$ *does not vanish at a fixed point* $p_1$ *of the involution* $g$ *and that* $g^* \omega = -\omega$.

*Let* $M$ *be any non-singular model of the quotient variety* $V/G$ *where* $G$ *is a cyclic group generated by* $g$. *Then, if* $|mK_M| \neq \phi$, $|mK_M|$ *has a fixed component*.

Proof. Let $p_1, p_2, \ldots, p_k$ be fixed points of the involution $g$. The quotient space $V/G$ has singular points $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\ldots, \mathfrak{p}_\ell$ which correspond to the fixed points. Each singular point has a neighbourhood which is analytically isomorphic to $\hat{U}$ in § 1. Let $M$ be a non-singular model of $V/G$ obtained by the canonical resolution of its singularities. First we shall show that if $|mK_M| \neq \emptyset$, then $|mK_M|$ has fixed components. Let $E_1, \ldots, E_k$ be subvarieties of $M$ appearing in the canonical resolution. From Lemma 1.2, ① we infer that there is an isomorphism

$$H^0(V, \underline{O}(mK_V))^G \xrightarrow{\sim} H^0(M, \underline{O}(mK_M))$$

and any element of $H^0(M, \underline{O}(mK_M))$ has zero of order at least $[\frac{m}{2}]$ on $E_i$. Hence the divisor $[\frac{m}{2}](E_1 + \cdots + E_k)$ is a fixed component of $|mK_M|$.

Next let us consider a birationally equivalent non-singular model $M^*$ of $M$. Let $g : M \longrightarrow M^*$ be a birational morphism. By elimination of the points of indeterminacy of a rational mapping due to Hironaka, there exist an algebraic manifold $\hat{M}$ and a

$$\begin{tikzcd} & \hat{M} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ M \arrow[rr, "g"] & & M^* \end{tikzcd}$$

morphism $\pi_1 : \hat{M} \longrightarrow M$ obtained by a finite succession of monoidal transformations with non-singular centers such that $\pi_2 = g \circ \pi_1 : \hat{M} \longrightarrow M^*$ is a morphism. Let $\mathcal{E}$ be the exceptional divisors appearing in the monoidal transformations. Then for any element $\varphi \longrightarrow H^0(M, \underline{O}(mK_M))$, $\pi_1^*(\varphi)$ has zeros on $\mathcal{E}$. Hence if $|mK_{\hat{M}}|$

$\neq \emptyset$, $|mK_{\hat{M}}|$ has a fixed component. We let $\hat{E}_i$, $i = 1,\ldots,k$ be the strict transform of $E_i$ to $\hat{M}$.

First we show that there exist at least one $\hat{E}_i$ or an irreducible component $\mathcal{E}_1$ of $\mathcal{E}$ such that $\pi_2(\hat{E}_i)$ or $\pi_2(\mathcal{E}_1)$ is a divisor on $M^*$. Assume the contrary. Then $\pi_2(E_i)$ and $\pi_2(\mathcal{E})$ are of codimension at least two in $M^*$. Let us consider the holomorphic 1-form $\omega$ on $M$. Since $M$ is algebraic, $\omega$ is a closed form. Hence we can choose a coordinate neighbourhood $U$ of the fixed point $p_1$ in $M$ with local coordinates $z_1, z_2, \ldots, z_n$ with center $p_1$ such that $\omega$ has a form $dz_1$ and that the involution is expressed in the form

$$(z_1, z_2, \ldots, z_n) \longrightarrow (-z_1, -z_2, \ldots, -z_n).$$

The form $(\omega)^2 \in H^0(V, \underline{S}^2(\Omega_V^1))^G$ induces a meromorphic section $\psi$ of $\underline{S}^2(\Omega_M^1)$ which is holomorphic on $M - \bigcup_{i=1}^{k} E_i$. Therefore the pull-back $\pi_1^*(\psi)$ is holomorphic on $\hat{M} - \bigcup_{i=1}^{k} \pi_1^{-1}(E_i)$. On the other hand if $S$ is the smallest analytic subset of $\hat{M}$ such that $\pi_2$ is an isomorphism on $\hat{M} - S$, then $\pi_2(S)$ is of codimension at least two by Zariski's Main Theorem. Hence $\pi_1^*(\psi)$ induces a holomorphic form on $M^* - \{\pi_2(\bigcup_{i=1}^{k} \pi_1^{-1}(E_i) \cup S)\}$. By our assumption $\pi_2(\pi_1^{-1}(E_i))$ is of codimension at least two. Since $\pi_2^*(g^*(\psi)) = \pi^*(\psi)$, $g^*(\psi)$ is holomorphic on $M^*$. Then by Lemma 1.1, $\psi$ must be holomorphic on $M$. But by Lemma 1.3, ②, $\psi$ has a pole on $E_1$. This is a contradiction. Hence

$\pi_2(E_i)$ or $\pi_2(\mathcal{E}_1)$ is a divisor. For simplicity we assume that $\pi_2(E)$ is a divisor. By Zariski's Main Theorem, there exists a nowhere dense algebraic subset $S$ such that $S \neq E_1$ and that at any point of $S-E_1$, $\pi_2$ is an isomorphism. Hence for any element $\varphi \in H^0(M, \underline{O}(mK_M))$ $g^*(\varphi)$ has a zero on $\pi_2(E_1)$. Since $\pi_2^*(g^*(\varphi)) = \pi_1^*(\varphi)$. By Lemma 1.1, if $|mK_{M^*}| \neq \emptyset$, then $|mK_{M^*}|$ has a fixed component $\pi_2(E_1)$. Q.E.D.

Remark 2.2. ① The above theorem holds for a compact complex manifold $V$ if we assume, furthermore, that a holomorphic 1-form $\omega$ in the above condition ② is d-closed.

② In the above theorem, if we assume that any fixed manifold of the involution $g$ is of codimension at least three and that there exists a holomorphic 1-form $\omega$ on $V$ such that $\omega$ has no zeros on a fixed manifold $F$, and that $\omega | F = 0$ and $g^*(\omega) = -\omega$, then the same conclusion holds.

§3. Now we prove Main Theorem. For simplicity we shall prove the theorem when $n = 3$.

(3.1) Let $C$ be a non-singular complete curve of genus $g$. Suppose that $C$ has an involution $\imath$ which has at least one fixed point. We set $\hat{C} = C/\langle\imath\rangle$. Assume that the genus of $\hat{C}$ is strictly greater than one. Let $S$ be a surface in $\mathbb{P}^3$ defined by the homogeneous equation

$$z_0^{10} + z_1^{10} + z_2^{10} + z_3^{10} = 0.$$

$S$ has an involution $h$ defined by

$$h : (z_0 : z_1 : z_2 : z_3) \longrightarrow (z_0 : -z_1 : -z_2 : z_3).$$

The involution $h$ has twenty fixed points on $S$. Let $\tilde{S}$ be a non-singular model of the quotient variety $S/\langle h \rangle$. Since there exists a surjective rational mapping of $\tilde{S}$ onto the surface $F$ in $\mathbb{P}^3$ defined by the homogeneous equation

$$z_0^5 + z_1^5 + z_2^5 + z_3^5 = 0,$$

we have $2 \geqq \kappa(\tilde{S}) \geqq \kappa(F) = 2$.

Let $g$ be an involution of $V = C \times S$ defined by

$$g : C \times S \longrightarrow C \times S$$
$$(z, w) \longmapsto (\iota(z), h(w)).$$

Since the canonical series $|K_C|$ of the curve $C$ has no base points, there exists a holomorphic 1-form $\omega$ on $C$ which does not vanish at a fixed point $p$ of $C$. We can consider $\omega$ as a holomorphic 1-form on $V$. Then the conditions (1) and (2) in Theorem 2.1 are satisfied. We let $M$ be the non-singular model of the quotient variety $V/\langle g \rangle$. By Theorem 2.1 $M$ satisfies the condition 2) in Main Theorem. Since there exists a surjective rational mapping of $M$ onto $\hat{C} \times \tilde{S}$, we have

$$3 \geqq \kappa(M) = \kappa(\hat{C} \times \tilde{S}) = \kappa(\hat{C}) + \kappa(S) = 3.$$

(3.2) Let $E$ be an elliptic curve. We set $V = E \times S$ where $S$ is the same as above. $V$ has an involution

$$g : E \times S \longrightarrow E \times S$$
$$(z, w) \longmapsto (-z, h(w))$$

where $h$ is the same involution as above. It is easy to show that $V$, $g$ and a holomorphic 1-form $\omega$ on $E$ satisfy the conditions in Theorem 2.1. Let $M$ be a non-singular model of $V/\langle g\rangle$ obtained by the canonical resolution of its singularity. Then $M$ satisfies the condition 2) in Main Theorem. There exists a surjective rational mapping $f : M \longrightarrow \tilde{S}$ whose general fibre is the elliptic curve $C$. Hence $f : M \longrightarrow \tilde{S}$ is birationally equivalent to an elliptic threefold. From the canonical bundle formula for elliptic threefolds (see Ueno [6], Theorem 6.1), we infer that $\kappa(M) = 2$.

(3.3) Let $C$, $\iota$ and $\omega$ be the same as those in 3.1. We let $T$ be an abelian surface. We set $V = C \times T$. $V$ has an involution $g$ defined by

$$\begin{aligned} g : C \times T &\longrightarrow C \times T \\ (z, w) &\longmapsto (\iota(z), -w). \end{aligned}$$

It is easy to show that $V$ and $g$ satisfy the conditions in Theorem 2.1. Let $M$ be a non-singular model of the quotient variety $V/\langle g\rangle$ obtained by the canonical resolution of its singularities. There exists a surjective rational mapping $f : M \longrightarrow \hat{C}$ whose general fibre is the abelian surface $S$. It is easy to calculate the canonical bundle formula of such a fibre space (see Ueno [8]) and we obtain

$$\kappa(M) = 1.$$

(3.4) Let $V$ be an abelian variety of dimension 3. $V$ has a natural involution

$$g : V \longrightarrow V$$
$$z \longmapsto -z \ .$$

A non-singular model $M$ of the quotient manifold $V/\langle g\rangle$ obtained by the canonical resolution of its singularities is usually called a Kummer manifold. $\kappa(M) = 0$ and $M$ satisfies the conditions of Main Theorem. Such a manifold has been studied in Ueno [7], §16.

Remark 3.5. Let $M$ be an algebraic threefold defined in 3.1. It is easy to show that $|mK_M| \neq \emptyset$ for any positive integer $m$. The m-th canonical mapping

$$\Phi_{mK} : M \longrightarrow \mathbb{P}^N$$

associated with the complete linear system $|mK|$ is a morphism. If $m$ is sufficiently large, the image $\Phi_{mK}(M)$ is analytically isomorphic to the quotient variety $V/\langle g\rangle$. Hence the image variety $\Phi_{mK}(M)$ is normal.

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